ESTIMATES FOR THE EXTINCTION TIME FOR THE RICCI FLOW ON CERTAIN 3–MANIFOLDS AND A QUESTION OF PERELMAN

TOBIAS H. COLDING AND WILLIAM P. MINICOZZI II

Abstract. We show that the Ricci flow becomes extinct in finite time on any Riemannian 3–manifold without aspherical summands.

1. introduction

In this note we prove some bounds for the extinction time for the Ricci flow on certain 3–manifolds. Our interest in this comes from a question that Grisha Perelman asked the first author at a dinner in New York City on April 25th of 2003. His question was "what happens to the Ricci flow on the 3–sphere when one starts with an arbitrary metric? In particular, does the flow become extinct in finite time?" He then went on to say that one of the difficulties in answering this is that he knew of no good way of constructing minimal surfaces for such a metric in general. However, there is a natural way of constructing such surfaces and that comes from the min–max argument where the minimal of all maximal slices of sweep–outs is a minimal surface; see, for instance, [2]. The idea is then to look at how the area of this min–max surface changes under the flow. Geometrically the area measures a kind of width of the 3–manifold and as we will see for certain 3–manifolds (those, like the 3–sphere, whose prime decomposition contains no aspherical factors) the area becomes zero in finite time corresponding to that the solution becomes extinct in finite time. Moreover, we will discuss a possible lower bound for how fast the area becomes zero. Very recently Perelman posted a paper (see [9]) answering his original question about finite extinction time. However, even after the appearance of his paper, then we still think that our slightly different approach may be of interest. In part because it is in some ways geometrically more natural, in part because it also indicates that lower bounds should hold, and in part because it avoids using the curve shortening flow that he simultaneously with the Ricci flow needed to invoke and thus our approach is in some respects technically easier.

Let M^3 be a smooth closed orientable 3–manifold and let $g(t)$ be a one–parameter family of metrics on M evolving by the Ricci flow, so

$$\partial_t g = -2 \operatorname{Ric}_{M_t} . \tag{1.1}$$

Unless otherwise stated we will assume throughout that M is prime and non–aspherical (so $\pi_k(M) \neq \{0\}$ for some $k > 1$). If M is prime but not irreducible, then $M = \mathbf{S}^2 \times \mathbf{S}^1$ (proposition 1.4 in [5]) so $\pi_3(M) = \mathbf{Z}$. Otherwise, if M is irreducible, then the sphere theorem implies that $\pi_2(M) = 0$ (corollary 3.9 in [5]). In the second

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case, the Hurewicz isomorphism theorem then implies that $\pi_3(M) \neq \{0\}$ (since M is non–aspherical). Therefore, in either case, by suspension, as in lemma 3 of [8], the space of maps from $\mathbf{S}^2$ to M is not simply connected.

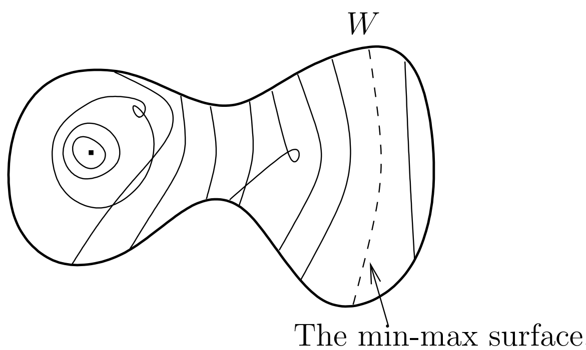

FIGURE 1. The sweep–out, the min–max surface, and the width W.

Fix a continuous map $\beta : [0,1] \to C^0 \cap L^2_1(\mathbf{S}^2, M)$ where $\beta(0)$ and $\beta(1)$ are constant maps so that β is in the nontrivial homotopy class $[\beta]$. We define the width $W = W(g, [\beta])$ by

$$W(g) = \min_{\gamma \in [\beta]} \max_{s \in [0,1]} \mathrm{E}(\gamma(s)) \,. \tag{1.2}$$

One could equivalently define the width using the area rather than the energy, but the energy is somewhat easier to work with. As for the Plateau problem, this equivalence follows using the uniformization theorem and the inequality $\mathrm{Area}(u) \leq \mathrm{E}(u)$ (with equality when u is a branched conformal map); cf. lemma 4.12 in [3].[1]

The next theorem gives an upper bound for the derivative of $W(g(t))$ under the Ricci flow which forces the solution $g(t)$ to become extinct in finite time (see paragraph 4.4 of [11] for the precise definition of extinction time when surgery occurs).

Theorem 1.1. *Let M^3 be a closed orientable prime non–aspherical 3–manifold equipped with a Riemannian metric $g = g(0)$. Under the Ricci flow, the width $W(g(t))$ satisfies*

$$\frac{d}{dt} W(g(t)) \leq -4\pi + \frac{3}{4(t+C)}\, W(g(t)) \,, \tag{1.3}$$

in the sense of the limsup of forward difference quotients. Hence, $g(t)$ must become extinct in finite time.

[1]It may be of interest to compare our notion of width, and the use of it, to a well–known approach to the Poincaré conjecture. This approach asks to show that for any metric on a homotopy 3–sphere a min–max type argument produces an embedded minimal 2–sphere. Note that in the definition of the width it plays no role whether the minimal 2–sphere is embedded or just immersed, and thus, the analysis involved in this was settled a long time ago. This well–known approach has been considered by many people, including Freedman, Meeks, Pitts, Rubinstein, Schoen, Simon, Smith, and Yau; see [2].

The 4π in (1.3) comes from the Gauss–Bonnet theorem and the $3/4$ comes from the bound on the minimum of the scalar curvature that the evolution equation implies. Both of these constants matter whereas the constant C depends on the initial metric and the actual value is not important.

To see that (1.3) implies finite extinction time rewrite (1.3) as

$$\frac{d}{dt}\left(W(g(t))\,(t+C)^{-3/4}\right) \leq -4\pi\,(t+C)^{-3/4} \tag{1.4}$$

and integrate to get

$$(T+C)^{-3/4}\,W(g(T)) \leq C^{-3/4}\,W(g(0)) - 16\,\pi\,\left[(T+C)^{1/4} - C^{1/4}\right]. \tag{1.5}$$

Since $W \geq 0$ by definition and the right hand side of (1.5) would become negative for T sufficiently large we get the claim.

Arguing as in 1.5 of [9] (or alternatively using Section 4), we get as a corollary of this theorem finite extinction time for the Ricci flow on all 3–manifolds without aspherical summands.

Corollary 1.2. *Let M^3 be a closed orientable 3–manifold whose prime decomposition has only non–aspherical factors and is equipped with a Riemannian metric $g = g(0)$. Under the Ricci flow with surgery, $g(t)$ must become extinct in finite time.*

Part of Perelman's interest in the question about finite time extinction comes from the following: If one is interested in geometrization of a homotopy three-sphere (or, more generally, a three-manifold without aspherical summands) and knew that the Ricci flow became extinct in finite time, then one would not need to analyze what happens to the flow as time goes to infinity. Thus, in particular, one would not need collapsing arguments.

2. Upper bounds for the rate of change of area of minimal 2–spheres

Suppose that $\Sigma \subset M$ is a closed immersed surface (not necessarily minimal), then using (1.1) an easy calculation gives (cf. page 38–41 of [4])

$$\frac{d}{dt}_{t=0}\text{Area}_{g(t)}(\Sigma) = -\int_\Sigma [R - \text{Ric}_M(\mathbf{n},\mathbf{n})]\,. \tag{2.1}$$

If Σ is also minimal, then

$$\begin{aligned}\frac{d}{dt}_{t=0}\text{Area}_{g(t)}(\Sigma) &= -2\int_\Sigma \text{K}_\Sigma - \int_\Sigma [|A|^2 + \text{Ric}_M(\mathbf{n},\mathbf{n})] \\ &= -\int_\Sigma K_\Sigma - \frac{1}{2}\int_\Sigma [|A|^2 + R]\,.\end{aligned} \tag{2.2}$$

Here K_Σ is the (intrinsic) curvature of Σ, $\mathbf{n}$ is a unit normal for Σ (our Σ's below will be $\mathbf{S}^2$'s and hence have a well–defined unit normal), A is the second fundamental form of Σ so that $|A|^2$ is the sum of the squares of the principal curvatures, Ric_M is the Ricci curvature of M, and R is the scalar curvature of M. (The curvature is normalized so that on the unit $\mathbf{S}^3$ the Ricci curvature is 2 and the scalar curvature is 6.) To get (2.2), we used that by the Gauss equations and minimality of Σ

$$\text{K}_\Sigma = \text{K}_M - \frac{1}{2}|A|^2\,, \tag{2.3}$$

where K_M is the sectional curvature of M on the two–plane tangent to Σ.

Our first lemma gives an upper bound for the rate of change of area of minimal 2–spheres.

Lemma 2.1. *If $\Sigma \subset M^3$ is a branched minimal immersion of the 2–sphere, then*

$$\frac{d}{dt}_{t=0} Area_{g(t)}(\Sigma) \leq -4\pi - \frac{Area_{g(0)}(\Sigma)}{2} \min_M R(0) \, . \tag{2.4}$$

Proof. Let $\{p_i\}$ be the set of branch points of Σ and $b_i > 0$ the order of branching at p_i. By (2.2)

$$\frac{d}{dt}_{t=0} \text{Area}_{g(t)}(\Sigma) \leq - \int_\Sigma K_\Sigma - \frac{1}{2} \int_\Sigma R = -4\pi - 2\pi \sum b_i - \frac{1}{2} \int_\Sigma R \, , \tag{2.5}$$

where the equality used the Gauss–Bonnet theorem with branch points. Note that branch points only help in the inequality (2.4). □

For an immersed minimal surface $\Sigma \subset M$ we set

$$L_\Sigma \, \phi = \Delta_\Sigma \, \phi + |A|^2 \, \phi + \text{Ric}_M(\mathbf{n}, \mathbf{n}) \, \phi \, . \tag{2.6}$$

By (2.2) and the Gauss–Bonnet theorem

$$\frac{d}{dt}_{t=0} \text{Area}_{g(t)}(\Sigma) = -2 \int_\Sigma \text{K}_\Sigma - \int_\Sigma 1 \cdot L_\Sigma \, 1 = -4\pi\chi(\Sigma) - \int_\Sigma 1 \cdot L_\Sigma \, 1 \, . \tag{2.7}$$

(Note that by the second variational formula (see, for instance, section 1.7 of [3]), then

$$\frac{\partial^2}{\partial r^2}_{r=0} \text{Area}(\Sigma_r) = - \int_\Sigma \phi \, L_\Sigma \, \phi \, , \tag{2.8}$$

where $\Sigma_r = \{x + r \, \phi(x) \, \mathbf{n}_\Sigma(x) \, | \, x \in \Sigma\}$.) Recall also that by definition the index of a minimal surface Σ is the number of negative eigenvalues (counted with multiplicity) of L_Σ. (A function η is an eigenfunction of L_Σ with eigenvalue λ if $L_\Sigma \, \eta + \lambda \, \eta = 0$.) Thus in particular, since Σ is assumed to be closed, the index is always finite.

3. Extinction in finite time

We begin by recalling a result on harmonic maps which gives the existence of minimal spheres realizing the width $W(g)$. The results of Sacks and Uhlenbeck give the harmonic maps but potentially allow some loss of energy. This energy loss was ruled out by Siu and Yau (using also arguments of Meeks and Yau), see Chapter VIII in [13]. For our purposes, the most convenient statement of this is given in theorem 4.2.1 of [6]. (The bound for the index is not stated explicitly in [6] but follows immediately as in [8].)

Proposition 3.1. *Given a metric g on M and a nontrivial $[\beta] \in \pi_1(C^0 \cap L_1^2(\mathbf{S}^2, M))$, there exists a sequence of sweep–outs $\gamma^j : [0,1] \to C^0 \cap L_1^2(\mathbf{S}^2, M)$ with $\gamma^j \in [\beta]$ so that*

$$W(g) = \lim_{j \to \infty} \max_{s \in [0,1]} E(\gamma^j_s) \, . \tag{3.1}$$

Furthermore, there exist $s_j \in [0,1]$ and branched conformal minimal immersions $u_0, \ldots, u_m : \mathbf{S}^2 \to M$ with index at most one so that, as $j \to \infty$, the maps $\gamma^j_{s_j}$

converge to u_0 *weakly in* L_1^2 *and uniformly on compact subsets of* $\mathbf{S}^2 \setminus \{x_1, \dots, x_k\}$, *and*

$$W(g) = \sum_{i=0}^{m} E(u_i) = \lim_{j \to \infty} E(\gamma^j_{s_j}) \, . \tag{3.2}$$

Finally, for each $i > 0$, *there exists a point* x_{k_i} *and a sequence of conformal dilations* $D_{i,j} : \mathbf{S}^2 \to \mathbf{S}^2$ *about* x_{k_i} *so that the maps* $\gamma^j_{s_j} \circ D_{i,j}$ *converge to* u_i.

Remark 3.2. It is implicit in Proposition 3.1 that $W(g) > 0$. This can, for instance, be seen directly using [6]. Namely, page 125 in [6] shows that if $\max_s \mathrm{E}(\gamma^j_s)$ is sufficiently small (depending on g), then γ^j is homotopically trivial.

We will also need a standard additional property for the min–max sequence of sweep–outs γ^j of Proposition 3.1 which can be achieved by modifying the sequence as in section 4 of [2] (cf. proposition 4.1 on page 85 in [2]). Loosely speaking this is the property that any subsequence $\gamma^k_{s_k}$ with energy converging to $W(g)$ converges (after possibly going to a further subsequence) to the union of branched immersed minimal 2–spheres each with index at most one. Precisely this is that we can choose γ^j so that: Given $\epsilon > 0$, there exist J and $\delta > 0$ (both depending on g and γ^j) so that if $j > J$ and

$$\mathrm{E}(\gamma^j_s) > W(g) - \delta \, , \tag{3.3}$$

then there is a collection of branched minimal 2–spheres $\{\Sigma_i\}$ each of index at most one and with

$$\mathrm{dist}\,(\gamma^j_s, \cup_i \Sigma_i) < \epsilon \, . \tag{3.4}$$

Here, the distance means varifold distance (see, for instance, section 4 of [2]). Below we will use that, as an immediate consequence of (3.4), if F is a quadratic form on M and Γ denotes γ^j_s, then

$$\left| \int_\Gamma [\mathrm{Tr}(F) - F(\mathbf{n}_\Gamma, \mathbf{n}_\Gamma)] - \sum_i \int_{\Sigma_i} [\mathrm{Tr}(F) - F(\mathbf{n}_{\Sigma_i}, \mathbf{n}_{\Sigma_i})] \right| < C \, \epsilon \, \|F\|_{C^1} \, \mathrm{Area}(\Gamma) \, . \tag{3.5}$$

In the proof of the result about finite extinction time we will also need that the evolution equation for $R = R(t)$, i.e. (see, for instance, page 16 of [4]),

$$\partial_t R = \Delta R + 2|\mathrm{Ric}|^2 \geq \Delta R + \frac{2}{3} R^2 \, , \tag{3.6}$$

implies by a straightforward maximum principle argument that at time $t > 0$

$$R(t) \geq \frac{1}{1/[\min R(0)] - 2t/3} = -\frac{3}{2(t + C)} \, . \tag{3.7}$$

In the derivation of (3.7) we implicitly assumed that $\min R(0) < 0$. If this was not the case, then (3.7) trivially holds with $C = 0$, since, by (3.6), $\min R(t)$ is always non–decreasing. This last remark is also used when surgery occurs. This is because by construction any surgery region has large (positive) scalar curvature.

Proof. (of Theorem 1.1) Fix a time τ. Below $\tilde{C}$ denotes a constant depending only on τ but will be allowed to change from line to line. Let $\gamma^j(\tau)$ be the sequence of sweep–outs for the metric $g(\tau)$ given by Proposition 3.1. We will use the sweep–out at time τ as a comparison to get an upper bound for the width at times $t > \tau$.

The key for this is the following claim (the inequality in (3.8) below): Given $\epsilon > 0$, there exist J and $\bar{h} > 0$ so that if $j > J$ and $0 < h < \bar{h}$, then

$$\begin{aligned} &\text{Area}_{g(\tau+h)}(\gamma_s^j(\tau)) - \max_s \text{E}_{g(\tau)}(\gamma_s^j(\tau)) \\ &\leq [-4\pi + \tilde{C}\,\epsilon + \frac{3}{4(\tau+C)} \max_s \text{E}_{g(\tau)}(\gamma_s^j(\tau))]\, h + \tilde{C}\, h^2 \,. \end{aligned} \tag{3.8}$$

To see why (3.8) implies (1.3), we use the definition of the width to get

$$W(g(\tau+h)) \leq \max_{s \in [0,1]} \text{Area}_{g(\tau+h)}(\gamma_s^j(\tau)) \,, \tag{3.9}$$

and then take the limit as $j \to \infty$ (so that $\max_s \text{E}_{g(\tau)}(\gamma_s^j(\tau)) \to W(g(\tau)))$ in (3.8) to get

$$\frac{W(g(\tau+h)) - W(g(\tau))}{h} \leq -4\pi + \tilde{C}\,\epsilon + \frac{3}{4(\tau+C)}\, W(g(\tau)) + \tilde{C}\, h \,. \tag{3.10}$$

Taking $\epsilon \to 0$ in (3.10) gives (1.3).

It remains to prove (3.8). First, let $\delta > 0$ and J, depending on ϵ (and on τ), be given by (3.3)–(3.5). If $j > J$ and $\text{E}_{g(\tau)}(\gamma_s^j(\tau)) > W(g) - \delta$, then let $\cup_i \Sigma_{s,i}^j(\tau)$ be the collection of minimal spheres in (3.5). Combining (2.1), (3.5) with $F = \text{Ric}_M$, and Lemma 2.1 gives

$$\begin{aligned} \frac{d}{dt}_{t=\tau} \text{Area}_{g(t)}(\gamma_s^j(\tau)) &\leq \frac{d}{dt}_{t=\tau} \text{Area}_{g(t)}(\cup_i \Sigma_{s,i}^j(\tau)) + \tilde{C}\,\epsilon\, \|\text{Ric}_M\|_{C^1}\, \text{Area}_{g(t)}(\gamma_s^j(\tau)) \\ &\leq -4\pi - \frac{\text{E}_{g(\tau)}(\gamma_s^j(\tau))}{2} \min_M R(\tau) + \tilde{C}\,\epsilon \\ &\leq -4\pi + \frac{3}{4(\tau+C)} \max_s \text{E}_{g(\tau)}(\gamma_s^j(\tau)) + \tilde{C}\,\epsilon \,, \end{aligned} \tag{3.11}$$

where the last inequality used the lower bound (3.7) for $R(\tau)$. Since the metrics $g(t)$ vary smoothly and every sweep–out γ^j has uniformly bounded energy, it is easy to see that $\text{E}_{g(\tau+h)}(\gamma_s^j(\tau))$ is a smooth function of h with a uniform C^2 bound independent of both j and s near $h = 0$ (cf. (2.1)). In particular, (3.11) and Taylor expansion gives $\bar{h} > 0$ (independent of j) so that (3.8) holds for s with $\text{E}_{g(\tau)}(\gamma_s^j(\tau)) > W(g) - \delta$. In the remaining case, we have $\text{E}(\gamma_s^j(\tau)) \leq W(g) - \delta$ so the continuity of $g(t)$ implies that (3.8) automatically holds after possibly shrinking $\bar{h} > 0$.

Finally, we claim that (1.3) implies finite extinction time. Namely, rewriting (1.3) as $\frac{d}{dt}\left(W(g(t))\,(t+C)^{-3/4}\right) \leq -4\pi\,(t+C)^{-3/4}$ and integrating gives

$$(T+C)^{-3/4}\, W(g(T)) \leq C^{-3/4}\, W(g(0)) - 16\,\pi\, \left[(T+C)^{1/4} - C^{1/4}\right] . \tag{3.12}$$

Since $W \geq 0$ by definition and the right hand side of (3.12) would become negative for T sufficiently large, the theorem follows. □

4. Remarks on the reducible case

When M is reducible, then the factors in the prime decomposition must split off in a uniformly bounded time. This follows from a (easy) modification of the proof of Theorem 1.1. Namely, each (non–trivial) factor in the prime decomposition gives rise to a 2–sphere which does not bound a 3–ball in M and, hence, to a stable minimal 2–sphere in this isotopy class by [7]. Applying the argument of the proof

of Theorem 1.1 to these minimal 2–spheres, we see that the minimal area in this isotopy class must go to zero in finite time as claimed.

Appendix A. Lower bounds for the rate of change of area of minimal 2–spheres

The next lower bound is an adaptation of Hersch's theorem; cf. [1]. Recall that Hersch's theorem (see, for instance, [12]) states the sharp scale invariant inequality that for any metric on the 2–sphere λ_1 times the area is bounded uniformly from above by the corresponding quantity on a round 2–sphere.

Lemma A.1. *If $\Sigma \subset M^3$ is an immersed minimal 2–sphere with index at most one, then*

$$8\,\pi \geq \int_\Sigma [|A|^2 + Ric_M(\mathbf{n},\mathbf{n})] = \int_\Sigma 1 \cdot L_\Sigma\, 1\,. \tag{A.1}$$

Hence, by (2.7)

$$\frac{d}{dt}_{t=0} Area_{g(t)}(\Sigma) \geq -16\,\pi\,. \tag{A.2}$$

Proof. If Σ is stable, i.e., if the index is zero, then for all ϕ

$$-\int_\Sigma \phi\, L\, \phi \geq 0\,, \tag{A.3}$$

or equivalently

$$\int_\Sigma |\nabla \phi|^2 \geq \int_\Sigma [|A|^2 + \mathrm{Ric}_M(\mathbf{n},\mathbf{n})]\, \phi^2\,, \tag{A.4}$$

and thus by letting $\phi \equiv 1$ in (A.4) we see that (A.1) holds.

If the index is one, then we let η be an eigenfunction for L_Σ with negative eigenvalue $\lambda < 0$. That is,

$$L_\Sigma\, \eta + \lambda\, \eta = 0\,. \tag{A.5}$$

By a standard argument, then an eigenfunction corresponding to the first eigenvalue of a Schrödinger operator (Laplacian plus potential) does not change sign and thus we may assume that η is everywhere positive. In particular, $\int_\Sigma \eta > 0$. Since Σ has index one then it follows that (A.4) holds for all ϕ with

$$0 = \int_\Sigma \eta\, \phi\,. \tag{A.6}$$

By the uniformization theorem, there exists a conformal diffeomorphism $\Phi : \Sigma \to \mathbf{S}^2 \subset \mathbf{R}^3$. For $i = 1, 2, 3$ set $\phi_i = x_i \circ \Phi$. For $x \in \mathbf{S}^2$ let $\pi_x : \mathbf{S}^2 \setminus \{x\} \to \mathbf{C}$ be the stereographic projection and let $\psi_{x,t}(y) = \pi_x^{-1}(t(\pi_x(y)))$, then for each t, x this can be extended to a conformal map on $\mathbf{S}^2$. Define $\Psi : \mathbf{S}^2 \times [0, 1) \to G$, where G is the group of conformal transformations of $\mathbf{S}^2$, by $\Psi(x, t) = \psi_{x,1/(1-t)}$. Since $\Psi(x, 0) = \mathrm{id}_{\mathbf{S}^2}$ for each $x \in \mathbf{S}^2$, Ψ can be thought of as a continuous map on $B_1(0) = \mathbf{S}^2 \times [0, 1)/(x, 0) \equiv (y, 0)$. Set

$$\mathcal{A}(\Psi(x,t)) = \frac{1}{\int_\Sigma \eta} \left(\int_\Sigma \eta\, x_i \circ \Psi(x,t) \circ \Phi \right)_{i=1,2,3}\,, \tag{A.7}$$

where η is as in (A.5). It follows that

$$\mathcal{A} : B_1(0) \to B_1(0) \text{ and } \lim_{(y,t)\to(x,1)} \mathcal{A}(\Psi(y,t)) = x\,. \tag{A.8}$$

In particular, it follows that $\mathcal{A}$ extends to $\partial B_1(0)$ as the identity map. We can therefore, by elementary topology (after possibly replacing Φ by $\psi \circ \Phi$), assume that for each i

$$\int_\Sigma \eta \, \phi_i = 0 \, ; \tag{A.9}$$

that is each ϕ_i is orthogonal to η. It follows from (A.4) that for each i

$$\int_\Sigma |\nabla \phi_i|^2 \geq \int_\Sigma [|A|^2 + \text{Ric}_M(\mathbf{n}, \mathbf{n})] \, \phi_i^2 \, . \tag{A.10}$$

Summing over i and using that $\Phi(\Sigma) \subset \mathbf{S}^2$ so $\sum_{i=1}^3 \phi_i^2 = 1$ we get

$$\sum_{i=1}^3 \int_\Sigma |\nabla \phi_i|^2 \geq \int_\Sigma [|A|^2 + \text{Ric}_M(\mathbf{n}, \mathbf{n})] \, . \tag{A.11}$$

Now obviously, since Φ is conformal (so that it preserves energy) and since each x_i is an eigenfunction for the Laplacian on $\mathbf{S}^2 \subset \mathbf{R}^3$ with eigenvalue $\lambda_1(\mathbf{S}^2) = 2$, we get

$$\int_\Sigma |\nabla \phi_i|^2 = \int_{\mathbf{S}^2} |\nabla x_i|^2 = \lambda_1(\mathbf{S}^2) \int_{\mathbf{S}^2} x_i^2 \, . \tag{A.12}$$

Combining (A.11) with (A.12) we get

$$2 \, \text{Area}(\mathbf{S}^2) = \sum_{i=1}^3 \lambda_1(\mathbf{S}^2) \int_{\mathbf{S}^2} x_i^2 = \sum_{i=1}^3 \int_{\mathbf{S}^2} |\nabla x_i|^2 \tag{A.13}$$

$$= \sum_{i=1}^3 \int_\Sigma |\nabla \phi_i|^2 \geq \int_\Sigma [|A|^2 + \text{Ric}_M(\mathbf{n}, \mathbf{n})] \, . \tag{A.14}$$

$\square$

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References

1. D. Christodoulou and S.T. Yau, *Some remarks on the quasi-local mass*, Mathematics and general relativity (Santa Cruz, CA, 1986), 9–14, Contemp. Math., vol. 71, Amer. Math. Soc., Providence, RI, 1988.
2. T.H. Colding and C. De Lellis, *The min–max construction of minimal surfaces*, Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), 75–107, Surv. Differ. Geom., VIII, Int. Press, Somerville, MA, 2003, math.AP/0303305.
3. T.H. Colding and W.P. Minicozzi II, *Minimal surfaces*, Courant Lecture Notes in Mathematics, 4. New York University, Courant Institute of Mathematical Sciences, New York, 1999.
4. R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, International Press, Cambridge, MA, 1995.
5. A. Hatcher, *Notes on basic* 3*–manifold topology*, www.math.cornell.edu/ hatcher/3M/3Mdownloads.html.
6. J. Jost, *Two–dimensional geometric variational problems*, J. Wiley and Sons, Chichester, N.Y. (1991).
7. W. Meeks III, L. Simon, and S.T. Yau, *Embedded minimal surfaces, exotic spheres and manifolds with positive Ricci curvature*, Ann. of Math. (2) **116** (1982) 621–659.
8. M.J. Micallef and J.D. Moore, *Minimal two–spheres and the topology of manifolds with positive curvature on totally isotropic two–planes*, Ann. of Math. (2) **127** (1988) 199–227.
9. G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three–manifolds*, math.DG/0307245.
10. G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, math.DG/0211159.

11. G. Perelman, *Ricci flow with surgery on three–manifolds*, math.DG/0303109.
12. R. Schoen and S.T. Yau, *Lectures on differential geometry*, International Press 1994.
13. R. Schoen and S.T. Yau, *Lectures on harmonic maps*, International Press 1997.

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012

Department of Mathematics, Johns Hopkins University, 3400 N. Charles St., Baltimore, MD 21218

E-mail address: colding@cims.nyu.edu and minicozz@math.jhu.edu